

A class of optimal ternary cyclic codes and their duals [☆]

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Abstract

Cyclic codes are a subclass of linear codes and have applications in consumer electronics, data storage systems, and communication systems as they have efficient encoding and decoding algorithms. Let $m = 2\ell + 1$ for an integer $\ell \geq 1$ and π be a generator of $\mathbb{GF}(3^m)^*$. In this paper, a class of cyclic codes $C_{(u,v)}$ over $\mathbb{GF}(3)$ with two nonzeros π^u and π^v is studied, where $u = (3^m + 1)/2$, and $v = 2 \cdot 3^\ell + 1$ is the ternary Welch-type exponent. Based on a result on the non-existence of solutions to certain equation over $\mathbb{GF}(3^m)$, the cyclic code $C_{(u,v)}$ is shown to have minimal distance four, which is the best minimal distance for any linear code over $\mathbb{GF}(3)$ with length $3^m - 1$ and dimension $3^m - 1 - 2m$ according to the Sphere Packing bound. The duals of this class of cyclic codes are also studied.

Keywords: Cyclic code, optimal code, sphere packing bound.

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1. Introduction

Let p be a prime. An $[n, k, d]$ linear code C over the finite field $\mathbb{GF}(p)$ is a k -dimensional subspace of $\mathbb{GF}(p)^n$ with minimum (Hamming) distance d , and is called cyclic if any cyclic shift of a codeword is another codeword of C . By identifying $(c_0, c_1, \dots, c_{n-1}) \in C$ with

$$c_0 + c_1x + c_2x^2 + \dots + c_{n-1}x^{n-1} \in \mathbb{GF}(p)[x]/(x^n - 1),$$

any cyclic code of length n over $\mathbb{GF}(p)$ corresponds to an ideal of the polynomial residue class ring $\mathbb{GF}(p)[x]/(x^n - 1)$. Note that every ideal of $\mathbb{GF}(p)[x]/(x^n - 1)$ is principal. Thus, any cyclic code C can be expressed as $C = \langle g(x) \rangle$, where $g(x)$ is monic and has the least degree. The polynomial $g(x)$ is called the generator polynomial and $h(x) = (x^n - 1)/g(x)$ is referred to as the parity-check polynomial of C . The cyclic code $C = \langle g(x) \rangle$ is said to have t nonzeros if its

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parity-check polynomial $h(x)$ can be factorized as a product of t distinct irreducible polynomials over $\mathbb{GF}(p)$ and accordingly the dual code C^\perp of C is said to have t zeros.

Let A_i denote the number of codewords with Hamming weight i in a code C of length n for $1 \leq i \leq n$. The weight enumerator of C is defined by

$$1 + A_1z + A_2z^2 + \cdots + A_nz^n,$$

and the vector $(1, A_1, A_2, \dots, A_n)$ is called the weight distribution of the code C . If C is a linear code, then the weight distribution of C gives the minimum distance and the error correcting capability of C . A code C is said to be a t -weight code if the number of nonzero A_i in the sequence (A_1, A_2, \dots, A_n) is equal to t [18].

Cyclic codes are a subclass of linear codes and have important applications in consumer electronics, data storage systems, and communication systems as they have efficient encoding and decoding algorithms compared with the linear block codes [13]. They also have applications in cryptography [1, 7] and sequence design [8]. During the past few decades, cyclic codes have received a lot of attention and much progress have been made (see [1], [3]-[6], [11], [14]-[17], [19], [20], [22]-[25] and the references therein).

Let $\mathbb{GF}(3^m)$ be the finite field with 3^m elements. Let π be a generator of $\mathbb{GF}(3^m)^*$ and $m_i(x)$ be the minimal polynomial of π^i over $\mathbb{GF}(3)$, where $\mathbb{GF}(3^m)^* = \mathbb{GF}(3^m) \setminus \{0\}$ and $0 \leq i \leq 3^m - 2$. Let $C_{(u,v)}$ be the cyclic code over $\mathbb{GF}(3)$ with generator polynomial $m_u(x)m_v(x)$, where u, v are two integers such that π^u and π^v are nonconjugate. When $u = 1$ and v is an integer such that x^v is a perfect nonlinear monomial, Carlet, Ding, and Yuan [1] proved that the code $C_{(1,v)}$ has parameters $[3^m - 1, 3^m - 1 - 2m, 4]$ which is optimal according to the Sphere Packing bound. Later, Ding and Helleseeth [6] constructed several classes of optimal ternary cyclic codes $C_{(1,v)}$ with parameters $[3^m - 1, 3^m - 1 - 2m, 4]$ using some monomials x^v over $\mathbb{GF}(3^m)$ including almost perfect nonlinear monomials. Zhou and Ding obtained a class of optimal ternary cyclic codes $C_{(u,v)}$ with parameters $[3^m - 1, 3^m - 1 - 2m, 4]$ by choosing $(u, v) = ((3^m + 1)/2, (3^k + 1)/2)$ where m is odd and k is even. Recently, Li *et al.* settled an open problem proposed by Ding and Helleseeth in [6] and obtained some classes of optimal ternary cyclic codes with parameters $[3^m - 1, 3^m - 1 - 2m, 4]$ and $[3^m - 1, 3^m - 2 - 2m, 5]$. The duals of the aforementioned optimal ternary cyclic codes are discussed in [10], [15], [21], [24], [25].

The objective of this paper is to study a class of ternary cyclic code $C_{(u,v)}$ with generator polynomial $m_u(x)m_v(x)$, where $u = (3^m + 1)/2$, and $v = 2 \cdot 3^\ell + 1$ is the ternary Welch-type exponent proposed by Dobbertin *et al.* [9] where they studied the cross-correlation between an m -sequence and its v -decimated version. Based on a result on the non-existence of solutions to certain equation over $\mathbb{GF}(3^m)$, the cyclic code $C_{(u,v)}$ is shown to have minimal distance four, which is the best minimal distance for such class of codes over $\mathbb{GF}(3)$. This family of cyclic codes is shown to have parameters $[3^m - 1, 3^m - 1 - 2m, 4]$ and is thus optimal according to the Sphere Packing bound. The duals of this class of cyclic codes are also studied.

2. An equation over $\mathbb{GF}(3^m)$

In this section, we study an equation over $\mathbb{GF}(3^m)$, where $m = 2\ell + 1$. The result on the non-existence of solutions to this equation will be used to determine the minimal distance of a class of cyclic codes in the sequel.

Lemma 2.1. *Let $m = 2\ell + 1$, where ℓ is a positive integer. Then for any given $\varepsilon \in \mathbb{GF}(3)^*$, the equation*

$$(x^{3^\ell} + \varepsilon)(x^{3^\ell} - x) = 1 \quad (1)$$

has no solution in $\mathbb{GF}(3^m)^$.*

Proof. It is clear that (1) holds if and only if

$$\begin{cases} x^{3^\ell} + \varepsilon = \theta \\ x^{3^\ell} - x = \frac{1}{\theta} \end{cases} \quad (2)$$

holds for some $\theta \in \mathbb{GF}(3^m)^*$. It is thus sufficient to prove that (2) does not have solutions in $\mathbb{GF}(3^m)^*$ for any $\theta \in \mathbb{GF}(3^m)^*$. Suppose on the contrary that (2) has a solution in $\mathbb{GF}(3^m)^*$ for some $\theta \in \mathbb{GF}(3^m)^*$, then θ should satisfy

$$\theta^{3^\ell} - \frac{1}{\theta^{3^\ell}} = \theta. \quad (3)$$

By (3), we immediately have

$$\left(\frac{\theta^2 - 1}{\theta} \right)^{3^\ell} = \theta,$$

which means that

$$(\theta^2 - 1)^{3^\ell} = \theta^{1+3^\ell}.$$

Thus $\theta^2 - 1$ is a square in $\mathbb{GF}(3^m)^*$. On the other hand, (3) holds for some $\theta \in \mathbb{GF}(3^m)^*$ if and only if

$$\theta^{3^\ell}(\theta^{3^\ell} - \theta) = 1.$$

Recall that $m = 2\ell + 1$. Taking 3 and $3^{\ell+1}$ powers on both sides of the equation above,

$$\theta^{3^{\ell+1}}(\theta^{3^{\ell+1}} - \theta^3) = 1, \quad (4)$$

$$\theta(\theta - \theta^{3^{\ell+1}}) = 1. \quad (5)$$

Using (4) and (5), we obtain

$$\frac{\theta^{3^{\ell+1}}(\theta^{3^{\ell+1}} - \theta^3) + 1}{(\theta(\theta - \theta^{3^{\ell+1}}) + 1)^2} - 1 = 1. \quad (6)$$

Note that

$$\begin{aligned} & \theta^{3^{\ell+1}}(\theta^{3^{\ell+1}} - \theta^3) + 1 - (\theta(\theta - \theta^{3^{\ell+1}}) + 1)^2 \\ = & (\theta^{3^{\ell+1}})^2 - \theta^3\theta^{3^{\ell+1}} + 1 - (\theta^2(\theta - \theta^{3^{\ell+1}})^2 + 2\theta(\theta - \theta^{3^{\ell+1}}) + 1) \\ = & ((\theta^{3^{\ell+1}})^2 + 2\theta\theta^{3^{\ell+1}} + \theta^2) - (\theta^2(\theta - \theta^{3^{\ell+1}})^2 + \theta^3\theta^{3^{\ell+1}}) \\ = & (\theta + \theta^{3^{\ell+1}})^2 - \theta^2(\theta + \theta^{3^{\ell+1}})^2 \\ = & (\theta + \theta^{3^{\ell+1}})^2(1 - \theta^2). \end{aligned}$$

Then (6) becomes

$$\left(\frac{\theta + \theta^{3^{\ell+1}}}{\theta(\theta - \theta^{3^{\ell+1}}) + 1} \right)^2 (1 - \theta^2) = 1.$$

Therefore, $1 - \theta^2$ is a square in $\mathbb{GF}(3^m)^*$. This is a contradiction since $\theta^2 - 1$ is a square in $\mathbb{GF}(3^m)^*$ and -1 is a nonsquare in $\mathbb{GF}(3^m)^*$ if m is odd. The proof of the lemma is finished. \square

3. A class of optimal ternary cyclic codes and their duals

In this section, suppose that $(u, v) = ((3^m + 1)/2, 2 \cdot 3^\ell + 1)$ for an odd integer $m = 2\ell + 1 \geq 3$. We shall study the properties of the cyclic code $C_{(u,v)}$ with two nonzeros π^u and π^v and its dual.

3.1. The parameters of cyclic code $C_{(u,v)}$

The length of the cyclic code $C_{(u,v)}$ is $3^m - 1$, and its dimension is determined by sizes of the cyclotomic cosets modulo $3^m - 1$ containing u and v . For $0 \leq j \leq 3^m - 2$. The cyclotomic coset modulo $3^m - 1$ containing j is defined as

$$C_j = \{j \cdot 3^s \pmod{3^m - 1} : s = 0, 1, 2, \dots, m - 1\}.$$

Theorem 3.1. *Let $m = 2\ell + 1$ and $(u, v) = ((3^m + 1)/2, 2 \cdot 3^\ell + 1)$. Then the code $C_{(u,v)}$ is an optimal ternary cyclic code with parameters $[3^m - 1, 3^m - 1 - 2m, 4]$.*

Proof. Note that $\gcd(u, 3^m - 1) = 2$ and $\gcd(v, 3^m - 1) = 1$ since m is odd. Then, it can be readily verified that $|C_u| = |C_v| = m$ and $C_u \cap C_v = \emptyset$. This implies that the dimension of $C_{(u,v)}$ is equal to $3^m - 1 - 2m$.

We now prove that the minimal distance d of $C_{(u,v)}$ is equal to 4. To this end, we first prove $d \geq 4$. It is clear that $d \geq 2$ since $c_i \pi^{iu} \neq 0$ for any $0 \leq i \leq 3^m - 2$ and $c_i \in \mathbb{GF}(3)^*$. By the definition of $C_{(u,v)}$, it has a codeword of Hamming weight 2 if and only if there exist two elements $c_1, c_2 \in \mathbb{GF}(3)^*$ and two distinct integers $0 \leq t_1 < t_2 \leq 3^m - 2$ such that

$$\begin{cases} c_1 \pi^{ut_1} + c_2 \pi^{vt_2} = 0 \\ c_1 \pi^{vt_1} + c_2 \pi^{ut_2} = 0. \end{cases} \quad (7)$$

Note that $\gcd(v, 3^m - 1) = 1$. It follows from the second equation of (7) that $c_1 = c_2$ and $t_2 = t_1 + (3^m - 1)/2$ since $t_1 \neq t_2$. Then the first equation becomes $2c_1 \pi^{ut_1} = 0$, which is impossible. Thus the code $C_{(u,v)}$ does not have a codeword of Hamming weight 2. We continue the proof to show that $C_{(u,v)}$ has no codewords of weight 3. Otherwise, there exist three elements c_1, c_2, c_3 in $\mathbb{GF}(3)^*$ and three distinct integers $0 \leq t_1 < t_2 < t_3 \leq 3^m - 2$ such that

$$\begin{cases} c_1 \pi^{ut_1} + c_2 \pi^{ut_2} + c_3 \pi^{ut_3} = 0 \\ c_1 \pi^{vt_1} + c_2 \pi^{vt_2} + c_3 \pi^{vt_3} = 0. \end{cases} \quad (8)$$

Let $x_i = \pi^{t_i}$ for $i = 1, 2, 3$. Then $x_1, x_2, x_3 \in \mathbb{GF}(3^m)^*$ and are distinct, and (8) becomes

$$\begin{cases} c_1 x_1^u + c_2 x_2^u + c_3 x_3^u = 0 \\ c_1 x_1^v + c_2 x_2^v + c_3 x_3^v = 0. \end{cases} \quad (9)$$

Let $y_1 = x_1/x_3$ and $y_2 = x_2/x_3$. It then follows from (9) that

$$\begin{cases} c_1 y_1'' + c_2 y_2'' + c_3 = 0 \\ c_1 y_1^v + c_2 y_2^v + c_3 = 0. \end{cases} \quad (10)$$

We only need to consider the solutions of (10) for $y_1, y_2 \in \mathbb{GF}(3^m)^* \setminus \{1\}$, since x_1, x_2, x_3 are pairwise distinct. Due to symmetry it is sufficient to consider the following two cases.

Case A: when $c_1 = c_2 = c_3 = 1$: In this case, we have

$$\begin{cases} y_1'' + y_2'' + 1 = 0 \\ y_1^v + y_2^v + 1 = 0. \end{cases} \quad (11)$$

Recall that $u = (3^m + 1)/2$. We have $y'' = y$ if y is a square in $\mathbb{GF}(3^m)^*$ and otherwise $y'' = -y$. We distinguish among the following four cases to prove that (11) cannot hold for any $y_1, y_2 \in \mathbb{GF}(3^m)^* \setminus \{1\}$.

(1) y_1, y_2 are squares in $\mathbb{GF}(3^m)^*$. In this subcase, (11) becomes

$$\begin{cases} y_1 + y_2 + 1 = 0 \\ y_1^v + y_2^v + 1 = 0 \end{cases}$$

which leads to

$$(1 + y_1)^v = 1 + y_1^v. \quad (12)$$

Notice that

$$(1 + y_1)^v = 1 + y_1^v + y_1^{2 \cdot 3^\ell} - y_1^{3^\ell + 1} - y_1^{3^\ell} + y_1.$$

This together with (12) yields

$$(y_1^{3^\ell} - y_1)(y_1^{3^\ell} - 1) = 0$$

which implies that $y_1 \in \mathbb{GF}(3)$ since $\gcd(m, \ell) = \gcd(2\ell + 1, \ell) = 1$. Thus $y_1 = -1$ since $y_1 \neq 0$ and $y_1 \neq 1$. This is a contradiction with the assumption that y_1 is a square in $\mathbb{GF}(3^m)^*$.

(2) y_1 is a square in $\mathbb{GF}(3^m)^*$ and y_2 is a nonsquare in $\mathbb{GF}(3^m)^*$. By a similar routine calculation as case (1), we arrive at

$$(y_1^{3^\ell} - y_1)(y_1^{3^\ell} - 1) = -(1 + y_1)^{2 \cdot 3^\ell + 1}.$$

Set $z_1 = 1 + y_1$. Then we have

$$(z_1^{3^\ell} - z_1)(z_1^{3^\ell} + 1) = -z_1^{2 \cdot 3^\ell + 1}.$$

Let $\tilde{z}_1 = z_1^{-1}$. Then dividing by $z_1^{2 \cdot 3^\ell + 1}$ on both sides of the equation above gives

$$(\tilde{z}_1^{3^\ell} - \tilde{z}_1)(\tilde{z}_1^{3^\ell} + 1) = 1. \quad (13)$$

By Lemma 2.1, (13) cannot hold for any $\tilde{z}_1 \in \mathbb{GF}(3^m)^*$.

(3) y_1 is a nonsquare in $\mathbb{GF}(3^m)^*$ and y_2 is a nonsquare in $\mathbb{GF}(3^m)^*$. This case is similar to case (2).

(4) y_1 and y_2 are nonsquares in $\mathbb{GF}(3^m)^*$. With a similar routine calculation as case (1), we have

$$(y_1^{3^\ell} + 1)(y_1^{3^\ell} - y_1) = 1.$$

This equation has no solutions in $\mathbb{GF}(3^m)^*$ due to Lemma 2.1.

Case B: when $c_1 = c_2 = 1$ and $c_3 = -1$. The proof of this case is similar to Case A. We omit the details here.

The discussion above shows that $d \geq 4$. On the other hand, according to the Sphere Packing bound (c.f., p. 48, [12]), the minimal distance of any linear code with length $3^m - 1$ and dimension $3^m - 1 - 2m$ should be less than or equal 4. Hence $d = 4$. This completes the proof. \square

3.2. The weights of the dual of $C_{(u,v)}$

In this section, we shall determine all the possible Hamming weights of the duals of $C_{(u,v)}$. Using Delsarte's Theorem [2], the dual of $C_{(u,v)}$ is given by

$$C_{(u,v)}^\perp = \{\mathbf{c}(a, b) : a, b \in \mathbb{GF}(3^m)^2\}$$

where the codeword

$$\mathbf{c}(a, b) = (\text{Tr}(a\pi^{-ui} + b\pi^{-vi}))_{i=0}^{3^m-2}$$

and Tr denotes the absolute trace from $\mathbb{GF}(3^m)$ to $\mathbb{GF}(3)$.

Theorem 3.2. *The weight of the codeword $\mathbf{c}(a, b)$ in $C_{(u,v)}^\perp$ is 0 if $a = b = 0$, and otherwise takes values from*

$$\{2 \cdot 3^{m-1}, 2 \cdot 3^{m-1} \pm 2 \cdot 3^\ell, 2 \cdot 3^{m-1} \pm 3^\ell\}.$$

Proof. Let χ_1 and χ denote the canonical additive character of $\mathbb{GF}(3)$ and $\mathbb{GF}(3^m)$, respectively. In terms of exponential sums, the weight of the codeword $\mathbf{c}(a, b) = (c_0, c_1, \dots, c_{3^m-2})$ in $C_{(u,v)}^\perp$ is given by

$$\begin{aligned} \text{WT}(\mathbf{c}(a, b)) &= \#\{0 \leq i \leq 3^m - 2 : c_i \neq 0\} \\ &= 3^m - 1 - \frac{1}{3} \sum_{i=0}^{3^m-2} \sum_{y \in \mathbb{GF}(3)} \chi_1(y c_i) \\ &= 3^m - 1 - \frac{1}{3} \sum_{i=0}^{3^m-2} \sum_{y \in \mathbb{GF}(3)} \chi_1(y (\text{Tr}(a\pi^{-ui}) + \text{Tr}(b\pi^{-vi}))) \\ &= 3^m - 1 - \frac{1}{3} \sum_{x \in \mathbb{GF}(3^m)^*} \sum_{y \in \mathbb{GF}(3)} \chi_1(ayx^u + byx^v) \\ &= 2 \cdot 3^{m-1} - \frac{1}{3} \sum_{y \in \mathbb{GF}(3)^*} \sum_{x \in \mathbb{GF}(3^m)} \chi_1(ayx^u + byx^v). \end{aligned} \tag{14}$$

Note that $x^u = x$ if x is a square in $\mathbb{GF}(3^m)$ and otherwise $x^u = -x$. It then follows from (14) that

$$\text{WT}(\mathbf{c}(a, b)) = 2 \cdot 3^{m-1} - \frac{1}{3} \sum_{y \in \mathbb{GF}(3)^*} (1 + \sum_{x \in \text{SQ}} \chi_1(ayx + byx^v) + \sum_{x \in \text{NSQ}} \chi_1(-ayx + byx^v)),$$

which leads to

$$\begin{aligned}
& 3\text{WT}(\mathbf{c}(a, b)) - 2 \cdot 3^m \\
&= 2 + \sum_{x \in \text{SQ}} \chi(ax + bx^v) + \sum_{x \in \text{SQ}} \chi(-ax - bx^v) + \sum_{x \in \text{NSQ}} \chi(-ax + bx^v) + \sum_{x \in \text{NSQ}} \chi(ax - bx^v) \\
&= 2 + \sum_{x \in \text{SQ}} \chi(ax + bx^v) + \sum_{x \in \text{NSQ}} \chi(ax + bx^v) + \sum_{x \in \text{NSQ}} \chi(-ax + bx^v) + \sum_{x \in \text{SQ}} \chi(-ax + bx^v) \\
&= \sum_{x \in \mathbb{GF}(3^m)} \chi(ax + bx^v) + \sum_{x \in \mathbb{GF}(3^m)} \chi(-ax + bx^v), \tag{15}
\end{aligned}$$

where SQ and NSQ denote the set of all squares and nonsquares in $\mathbb{GF}(3^m)^*$ respectively, and the third equality follows from the fact that $-x$ runs through NSQ or SQ as x ranges over SQ or NSQ respectively.

We distinguish among the following three cases to determine all the possible weights of the codeword $\mathbf{c}(a, b)$.

Case A: when $a = b = 0$: It is clear that $\text{WT}(\mathbf{c}(a, b)) = 0$ in this case.

Case B: when $a = 0$ or $b = 0$: In this case, by (15) and the fact that x^v is a permutation over $\mathbb{GF}(3^m)$, we immediately obtain $\text{WT}(\mathbf{c}(a, b)) = 2 \cdot 3^{m-1}$.

Case C: when $a \neq 0$ and $b \neq 0$: It follows from (15) that

$$\text{WT}(\mathbf{c}(a, b)) = 2 \cdot 3^{m-1} - \frac{1}{3}(\hat{f}_v(\lambda) + \hat{f}_v(-\lambda)) \tag{16}$$

where

$$\hat{f}_v(\lambda) = \sum_{x \in \mathbb{GF}(3^m)} \chi(x^v - \lambda x)$$

is the fourier transform of the power function x^v at the point $\lambda = ab^{-1}$. It has been proven in [9] that $\hat{f}_v(\lambda) \in \{0, \pm 3^{\ell+1}\}$ for each $\lambda \in \mathbb{GF}(3^m)^*$. This together with (16) implies that $\text{WT}(\mathbf{c}(a, b))$ takes values from the set $\{2 \cdot 3^{m-1}, 2 \cdot 3^{m-1} \pm 2 \cdot 3^\ell, 2 \cdot 3^{m-1} \pm 3^\ell\}$.

The discussion above finishes the proof of this theorem. \square

3.3. Examples

The following are some examples for this class of cyclic codes and their duals, which are generated by a Magma program.

Example 3.3. Let $p = 3$, $m = 5$ and π be a generator of $\mathbb{GF}(3^m)^*$ with minimal polynomial $x^5 + 2x + 1$. Then the code $C_{(u,v)}$ is an optimal ternary cyclic code with generator polynomial $x^{10} + x^9 + 2x^8 + 2x^6 + 2x^5 + x^3 + 2x + 2$ and parameters $[242, 232, 4]$. The dual of the code $C_{(u,v)}$ has the following weight enumerator:

$$1 + 2420z^{144} + 12100z^{153} + 34364z^{162} + 7744z^{171} + 2420z^{180}.$$

Example 3.4. Let $p = 3$, $m = 7$ and π be a generator of $\mathbb{GF}(3^m)^*$ with minimal polynomial $x^7 + 2x^2 + 1$. Then the code $C_{(u,v)}$ is an optimal ternary cyclic code with generator polynomial $x^{14} + 2x^{12} + x^{10} + x^9 + 2x^8 + 2x^7 + 2x^5 + x^3 + x^2 + x + 2$ and parameters $[2186, 2172, 4]$. The dual of the code $C_{(u,v)}$ has the following weight enumerator:

$$1 + 153020z^{1404} + 1040536z^{1431} + 2513900z^{1458} + 922492z^{1485} + 153020z^{1512}.$$

Example 3.5. Let $p = 3$, $m = 9$ and π be a generator of $\mathbb{GF}(3^m)^*$ with minimal polynomial $x^9 + 2x^3 + 2x^2 + x + 1$. Then the code $C_{(u,v)}$ is an optimal ternary cyclic code with generator polynomial $x^{18} + 2x^{17} + x^{16} + x^{14} + x^{12} + 2x^{11} + 2x^{10} + x^8 + 2x^6 + x^5 + 2x^4 + x^2 + x + 2$ and parameters $[19682, 19664, 4]$. The dual of the code $C_{(u,v)}$ has the following weight enumerator:

$$1 + 10628280z^{12960} + 88214724z^{13041} + 192922964z^{13122} + 85026240z^{13203} + 10628280z^{13284}.$$

4. Concluding Remarks

In this paper, a class of ternary cyclic codes and their duals were studied. Based on a result on the non-existence of solutions to an equation over $\mathbb{GF}(3^m)$, it was shown that this class of cyclic codes has parameters $[3^m - 1, 3^m - 1 - 2m, 4]$ for an odd integer m and thus is optimal with respect to certain bound on general linear code. The duals of this class of cyclic codes were shown to have at most five nonzero weights. It would be interesting if the weight distribution of the duals could be completely determined. The reader is invited to address this adventure.

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